eAppendix for *Bounding bias due to selection*

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1 General selection bias

Result 1A. Bounding factor

Assume that the causal risk ratio $\frac{P(Y_1=1)}{P(Y_0=1)}$ is identifiable (perhaps within strata of confounders) as

$$\mathrm{RR}_{AY}^{\text{true}} = \frac{P(Y=1|A=1)}{P(Y=1|A=0)} \ .$$

Assume, however, that we only have access to data in a selected sample, so we are actually estimating

$$\mathrm{RR}_{AY}^{\text{obs}} = \frac{P(Y=1|A=1,S=1)}{P(Y=1|A=0,S=1)} \ .$$

Finally, assume that although $Y_a \amalg A$, it is not the case that $Y_a \amalg A|S=1$, so that the observed risk ratio is a biased estimator of the true causal risk ratio in the total population. (When $Y_a \amalg A|S=1$, the causal risk ratio for the selected population is unbiased and can be estimated in the data, but may differ from the causal effect in the total population due to differences in the distribution of other risk factors for the outcome.)

We define the selection bias factor as

$$\text{bias} = \frac{\mathrm{RR}_{AY}^{\text{obs}}}{\mathrm{RR}_{AY}^{\text{true}}} \ .$$

Assume that bias > 1. If not, reverse the coding of A (so that we when we bound the bias from above, as follows, we are in fact bounding the originally coded bias from below).

Because

$$\mathrm{RR}^{\mathrm{true}}_{AY} = \frac{P(Y=1|A=1,S=0)P(S=0|A=1)+P(Y=1|A=1,S=1)P(S=1|A=1)}{P(Y=1|A=0,S=0)P(S=0|A=0)+P(Y=1|A=0,S=1)P(S=1|A=0)},$$

we have that

$$\begin{aligned}\text{bias} &\leq \left\{\frac{P(Y=1|A=1,S=1)}{P(Y=1|A=0,S=1)}\right\} \Big/ \left\{\frac{\min_s P(Y=1|A=1,S=s)}{\max_s P(Y=1|A=0,S=s)}\right\} \\ &= \left\{\frac{P(Y=1|A=1,S=1)}{\min_s P(Y=1|A=1,S=s)}\right\} \times \left\{\frac{\max_s P(Y=1|A=0,S=s)}{P(Y=1|A=0,S=1)}\right\}.\end{aligned} \tag{1}$$

We have 4 possibilities for the right-hand side, depending on what values S takes on to maximize and minimize the respective expressions.

Take first the case in which $S=0$ in both places. This occurs when $P(Y=1|A=1,S=1) \geq P(Y=1|A=1,S=0)$ and $P(Y=1|A=0,S=1) \leq P(Y=1|A=0,S=0)$.

Then

$$\text{bias} \leq \left\{\frac{P(Y=1|A=1,S=1)}{P(Y=1|A=1,S=0)}\right\} \times \left\{\frac{P(Y=1|A=0,S=0)}{P(Y=1|A=0,S=1)}\right\}. \tag{2}$$

Now assume that there exists U such that $Y \amalg S|\{A,U\}$. We will assume a categorical U with values $u=1,2,...,k$ for ease of notation, but U can also be continuous and/or a vector of random variables.

Since $P(Y=1|A=a,S=1,U=u) = P(Y=1|A=a,S=0,U=u) = P(Y=1|A=a,U=u)$, we can rewrite equation (2):

$$\begin{aligned}\text{bias} \leq &\left\{\frac{\sum_{u=1}^{k} P(Y=1|A=1,U=u)P(U=u|A=1,S=1)}{\sum_{u=1}^{k} P(Y=1|A=1,U=u)P(U=u|A=1,S=0)}\right\} \times \\ &\left\{\frac{\sum_{u=1}^{k} P(Y=1|A=0,U=u)P(U=u|A=0,S=0)}{\sum_{u=1}^{k} P(Y=1|A=0,U=u)P(U=u|A=0,S=1)}\right\}.\end{aligned}$$

By Lemma A.3. in Ding and VanderWeele 2016a,[1] we have that

$$\frac{\sum_{u=1}^{k} P(Y=1|A=1,U=u)P(U=u|A=1,S=1)}{\sum_{u=1}^{k} P(Y=1|A=1,U=u)P(U=u|A=1,S=0)} \leq \frac{\mathrm{RR}_{UY|(A=1)} \times \mathrm{RR}_{SU|(A=1)}}{\mathrm{RR}_{UY|(A=1)} + \mathrm{RR}_{SU|(A=1)} - 1} \tag{3}$$

and

$$\frac{\sum_{u=1}^{k} P(Y=1|A=0,U=u)P(U=u|A=0,S=0)}{\sum_{u=1}^{k} P(Y=1|A=0,U=u)P(U=u|A=0,S=1)} \leq \frac{\mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{SU|(A=0)}}{\mathrm{RR}_{UY|(A=0)} + \mathrm{RR}_{SU|(A=0)} - 1} \tag{4}$$

where

$$\mathrm{RR}_{UY|(A=1)} = \frac{\max_u P(Y=1|A=1,u)}{\min_u P(Y=1|A=1,u)}$$

$$\mathrm{RR}_{UY|(A=0)} = \frac{\max_u P(Y=1|A=0,u)}{\min_u P(Y=1|A=0,u)}$$

$$\mathrm{RR}_{SU|(A=1)} = \max_u \frac{P(U=u|A=1,S=1)}{P(U=u|A=1,S=0)}$$

$$\mathrm{RR}_{SU|(A=0)} = \max_u \frac{P(U=u|A=0,S=0)}{P(U=u|A=0,S=1)} \; .$$

These values can be interpreted as the maximum relative risks comparing any two values of U on Y within strata of $A=1$ and $A=0$, respectively; and the maximum factors by which selection increases the prevalence of some value of U within the stratum $A=1$ and by which non-selection increases the relative prevalence of some value of U within stratum $A=0$.

Each expression in the left-hand side of (2) is positive, so from (3) and (4) we have that

$$\text{bias} \leq \left(\frac{\mathrm{RR}_{UY|(A=1)} \times \mathrm{RR}_{SU|(A=1)}}{\mathrm{RR}_{UY|(A=1)} + \mathrm{RR}_{SU|(A=1)} - 1} \right) \times \left(\frac{\mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{SU|(A=0)}}{\mathrm{RR}_{UY|(A=0)} + \mathrm{RR}_{SU|(A=0)} - 1} \right) \; . \tag{5}$$

Now consider the cases in which S, in one of both of the expressions in (1), takes on the value 1. In that case, one or both of the factors in (1) is equal to 1.

If $P(Y=1|A=1,S=1) \leq P(Y=1|A=1,S=0)$ and $P(Y=1|A=0,S=1) \geq P(Y=1|A=0,S=0)$ then

$$\text{bias} \; \leq 1 \; .$$

If $P(Y=1|A=1,S=1) \geq P(Y=1|A=1,S=0)$ and $P(Y=1|A=0,S=1) \geq P(Y=1|A=0,S=0)$ then

$$\text{bias} \; \leq \frac{\mathrm{RR}_{UY|(A=1)} \times \mathrm{RR}_{SU|(A=1)}}{\mathrm{RR}_{UY|(A=1)} + \mathrm{RR}_{SU|(A=1)} - 1} \; .$$

If $P(Y=1|A=1,S=1) \leq P(Y=1|A=1,S=0)$ and $P(Y=1|A=0,S=1) \leq P(Y=1|A=0,S=0)$ then

$$\text{bias} \; \leq \frac{\mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{SU|(A=0)}}{\mathrm{RR}_{UY|(A=0)} + \mathrm{RR}_{SU|(A=0)} - 1} \; .$$

Because the right-hand side of equation (5) is greater or equal to the right-hand side of the three bias inequalities under the other three conditions, then it is an upper bound for the bias in each case.

Result 1B. Summary measure

To construct a summary measure for the strength of a given risk ratio against selection bias, we can find the smallest risk ratio implied by the bounding factor that would be sufficient to reduce a given $\mathrm{RR}_{AY}^{\mathrm{obs}}$ to $\mathrm{RR}_{AY}^{\mathrm{true}} = 1$, assuming each of the parameters in the bounding factor were of that same magnitude. Denote that value RR. Then

$$\mathrm{RR}_{AY}^{\mathrm{obs}} \leq \frac{\mathrm{RR}^4}{(2\mathrm{RR} - 1)^2} \ .$$

Solving this inequality for RR shows us that for selection bias to completely explain away $\mathrm{RR}_{AY}^{\mathrm{obs}}$,

$$\mathrm{RR}_{UY|(A=1)} = \mathrm{RR}_{UY|(A=0)} = \mathrm{RR}_{SU|(A=1)} = \mathrm{RR}_{SU|(A=0)} \geq \sqrt{\mathrm{RR}_{AY}^{\mathrm{obs}}} + \sqrt{\mathrm{RR}_{AY}^{\mathrm{obs}} - \sqrt{\mathrm{RR}_{AY}^{\mathrm{obs}}}} \ .$$

2 When $S = U$

Result 2A. Bounding factor

In some cases selection may be directly determined by U, so that $S = U$. Then $\mathrm{RR}_{SU|(A=0)} = \mathrm{RR}_{SU|(A=1)} = \frac{1}{0}$. To bound the bias in such cases we can take the limit of the right-hand side of equation (5) as each RR_{SU} approaches ∞:

$$\begin{aligned} \text{bias} \ &\leq \lim_{\mathrm{RR}_{SU} \to \infty} \left(\frac{\mathrm{RR}_{UY|(A=1)} \times \mathrm{RR}_{SU}}{\mathrm{RR}_{UY|(A=1)} + \mathrm{RR}_{SU} - 1} \right) \times \left(\frac{\mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{SU}}{\mathrm{RR}_{UY|(A=0)} + \mathrm{RR}_{SU} - 1} \right) \\ &= \mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{UY|(A=1)} \end{aligned}$$

Result 2B. Summary measure

When $S = U$, if $\mathrm{RR}_{AY}^{\mathrm{true}} = 1$, then

$$\mathrm{RR}_{AY}^{\mathrm{obs}} \leq \mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{UY|(A=1)} \ .$$

By the same reasoning as above, if we assume both parameters in the bounding factor are of the same magnitude, then

$$\mathrm{RR}_{UY|(A=0)} = \mathrm{RR}_{UY|(A=1)} \leq \sqrt{\mathrm{RR}_{AY}^{\mathrm{obs}}} \ .$$

3 Increased risk in both exposure groups

Result 3A. Bounding factor

When $P(Y = 1|A = 1, S = 1)/P(Y = 1|A = 1, S = 0)$ and $P(Y = 1|A = 0, S = 1)/P(Y = 1|A = 0, S = 0)$ are both greater than 1, equation (1) can be rewritten

$$\text{bias} \leq \frac{P(Y = 1|A = 1, S = 1)}{P(Y = 1|A = 1, S = 0)} \ .$$

Results 3A follows from the derivation of Result 1A using only that factor in (1), giving us:

$$\text{bias} \leq \frac{\mathrm{RR}_{UY|(A=1)} \times \mathrm{RR}_{SU|(A=1)}}{\mathrm{RR}_{UY|(A=1)} + \mathrm{RR}_{SU|(A=1)} - 1} \ .$$

Result 3B. Summary measure

Again denote by RR the smallest risk ratio implied by the bounding factor that would be sufficient to reduce a given $\text{RR}_{AY}^{\text{obs}}$ to $\text{RR}_{AY}^{\text{true}} = 1$, assuming each of the parameters in the bounding factor in Result 3B were of that same magnitude. Then

$$\text{RR}_{AY}^{\text{obs}} \leq \frac{\text{RR}^2}{2\text{RR} - 1} \ .$$

Solving this inequality for RR shows us that for selection bias to completely explain away $\text{RR}_{AY}^{\text{obs}}$,

$$\text{RR}_{UY|(A=1)} = \text{RR}_{SU|(A=1)} \geq \text{RR}_{AY}^{\text{obs}} + \sqrt{\text{RR}_{AY}^{\text{obs}} \left(\sqrt{\text{RR}_{AY}^{\text{obs}}} - 1 \right)} \ .$$

4 Increased risk in both exposure groups and when $S = U$

Result 4A. Bounding factor

Result 4A immediately follows from Results 2A and 3A.

Result 4B. Summary measure

Result 4B is trivial.

5 Inference in the selected population

Result 5A. Bounding factor

Now assume that the parameter of interest is the causal risk ratio within the selected population, $\frac{P(Y_1=1|S=1)}{P(Y_0=1|S=1)}$.

Again we can estimate

$$\text{RR}_{AY}^{\text{obs}} = \frac{P(Y=1|A=1, S=1)}{P(Y=1|A=0, S=1)}$$

from a sample, but since it is not the case that $Y_a \amalg A|S=1$, the observed risk ratio is again a biased estimator of the causal risk ratio.

Assume, however, that $Y_a \amalg A|\{S=1, U\}$, so that

$$\text{RR}_{AY|(S=1)}^{\text{true}} = \frac{\sum_{u=1}^{k} P(Y=1|A=1, U=u, S=1)P(U=u|S=1)}{\sum_{u=1}^{k} P(Y=1|A=0, U=u, S=1)P(U=u|S=1)}$$

By Result 1 in in Ding and VanderWeele 2016b,[2]

$$\frac{\text{RR}_{AY}^{\text{obs}}}{\text{RR}_{AY|(S=1)}^{\text{true}}} \leq \frac{\text{RR}_{UY|(S=1)} \times \text{RR}_{AU|(S=1)}}{\text{RR}_{UY|(S=1)} + \text{RR}_{AU|(S=1)} - 1}$$

where

$$\text{RR}_{UY|(S=1)} = \max_a \frac{\max_u P(Y=1|A=a, S=1, U=u)}{\min_u P(Y=1|A=a, S=1, U=u)}$$

$$\text{RR}_{AU|(S=1)} = \max_u \frac{P(U=u|A=1, S=1)}{P(U=u|A=0, S=1)} \ .$$

Result 5B. Summary measure

The analytic form of Result 5A is equivalent to that of Result 2A. It therefore follows that the minimum magnitude of each of the two parameters that make up the bounding factor in Result 5A, assuming they are equal, that would be sufficient to shift a given $\mathrm{RR}^{\mathrm{obs}}_{AY}$ to the null is given by:

$$\mathrm{RR}_{UY|(S=1)} = \mathrm{RR}_{AU|(S=1)} \leq \mathrm{RR}^{\mathrm{obs}}_{AY} + \sqrt{\mathrm{RR}^{\mathrm{obs}}_{AY}(\mathrm{RR}^{\mathrm{obs}}_{AY} - 1)}\ .$$

II Results for the risk difference

1 General selection bias

Result 1A. Bound

As with the risk ratio, we assume that the causal risk difference $P(Y_1 = 1) - P(Y_0 = 1)$ is identifiable as

$$\mathrm{RD}^{\mathrm{true}}_{AY} = P(Y = 1|A = 1) - P(Y = 1|A = 0)\ .$$

We exclude the variables necessary to eliminate confounding from the conditioning statement for ease of notation, but the above could hold conditional on confounders C, in which case assume all probability statements that follow are also conditional on confounders C.

If we only have data from a selected population, we observe

$$\mathrm{RD}^{\mathrm{obs}}_{AY} = P(Y = 1|A = 1, S = 1) - P(Y = 1|A = 0, S = 1)\ .$$

Again we assume that it is not the case that $Y_a \amalg A|S = 1$, so that $\mathrm{RD}^{\mathrm{obs}}_{AY}$ is a biased estimator of the causal risk difference. Now we are concerned with bias on the additive scale:

$$\text{bias } = \mathrm{RR}^{\mathrm{obs}}_{AY} - \mathrm{RR}^{\mathrm{true}}_{AY}\ .$$

Assume that the bias is non-negative; if not, recode the exposure A so that it is.

Because $\mathrm{RD}^{\mathrm{true}}_{AY} \geq \min_s P(Y = 1|A = 1, S = s) - \max_s P(Y = 1|A = 0, S = s)$, we have that

$$\begin{aligned}\text{bias } &\leq [P(Y = 1|A = 1, S = 1) - P(Y = 1|A = 0, S = 1)] \ - \\ &\quad \Big[\min_s P(Y = 1|A = 1, S = s) - \max_s P(Y = 1|A = 0, S = s)\Big]\ . \end{aligned} \tag{6}$$

The right-hand side of equation (6) is maximized with $S = 0$ in both conditioning statements, so we will find a bound for the bias under that condition.

We can therefore rewrite (6):

$$\begin{aligned}\text{bias } &\leq [P(Y = 1|A = 1, S = 1) - P(Y = 1|A = 1, S = 0)] \ + \\ &\quad [P(Y = 1|A = 0, S = 0) - P(Y = 1|A = 0, S = 1)]\ . \end{aligned} \tag{7}$$

The bias is bounded by the sum of two risk differences representing the association between S and Y within strata of A. To deal with each of them simultaneously we will consider bounding the apparent risk difference for any value $A = a$ and two values of S, s and s^*:

$$\mathrm{RD}^{\mathrm{app}}_{SY} = P(Y = 1|A = a, S = s) - P(Y = 1|A = a, S = s^*)\ . \tag{8}$$

(This risk difference is never actually observed because we have no data for the stratum $S = 0$, which must be either s or s^*.)

Assume there exists U such that $P(Y = 1|A = a, S = s, U = u) - P(Y = 1|A = a, S = s^*, U = u) = 0$ for all values u, or equivalently $Y \amalg S|\{A, U\}$. In other words, conditioning on U is sufficient to eliminate the apparent association between S and Y (and therefore the selection bias as well, as the extent to which $\mathrm{RD}_{SY}^{\mathrm{app}}$ is non-zero (for each value of A) is essentially the extent of the bias due to selection). We will denote the risk difference conditional on U as $\mathrm{RD}_{SY}^{\mathrm{true}}$.

Because $\mathrm{RD}_{SY}^{\mathrm{true}} = 0$, a bound for $\mathrm{RD}_{SY}^{\mathrm{app}} - \mathrm{RD}_{SY}^{\mathrm{true}}$ is also a bound for $\mathrm{RD}_{SY}^{\mathrm{app}}$.

We can use results from Ding and VanderWeele 2016b[2] to bound $\mathrm{RD}_{SY}^{\mathrm{app}}$. From their results we have that

$$\frac{P(Y = 1|A = a, S = s)}{P(Y = 1|A = a, S = s^*)} \leq \frac{\mathrm{RR}_{UY|(A=a)} \times \mathrm{RR}_{SU|(A=a)}}{\mathrm{RR}_{UY|(A=a)} + \mathrm{RR}_{SU|(A=a)} - 1} \tag{9}$$

where

$$\mathrm{RR}_{UY|(A=a)} = \frac{\max_u P(Y = 1|A = a, u)}{\min_u P(Y = 1|A = a, u)}$$

and

$$\mathrm{RR}_{SU|(A=a)} = \max_u \frac{P(U = u|A = a, S = s)}{P(U = u|A = a, S = s^*)} \ .$$

Rearranging (9) shows us that

$$\mathrm{RD}_{SY}^{\mathrm{app}} \leq P(Y = 1|A = a, S = s^*) \times \frac{\mathrm{RR}_{UY|(A=a)} \times \mathrm{RR}_{SU|(A=a)}}{\mathrm{RR}_{UY|(A=a)} + \mathrm{RR}_{SU|(A=a)} - 1} \ - \\ P(Y = 1|A = a, S = s) \times \frac{\mathrm{RR}_{UY|(A=a)} + \mathrm{RR}_{SU|(A=a)} - 1}{\mathrm{RR}_{UY|(A=a)} \times \mathrm{RR}_{SU|(A=a)}} \ .$$

Returning to equation (7), we now can replace each of the apparent risk differences with their bounds, which will be an overall bound for the bias:

$$\text{bias} \ \leq P(Y = 1|A = 1, S = 0) \times \mathrm{BF}_1 - P(Y = 1|A = 1, S = 1)/\mathrm{BF}_1 \ + \\ P(Y = 1|A = 0, S = 1) \times \mathrm{BF}_0 - P(Y = 1|A = 0, S = 0)/\mathrm{BF}_0$$

where

$$\mathrm{BF}_1 = \frac{\mathrm{RR}_{UY|(A=1)} \times \mathrm{RR}_{SU|(A=1)}}{\mathrm{RR}_{UY|(A=1)} + \mathrm{RR}_{SU|(A=1)} - 1}$$

and

$$\mathrm{BF}_0 = \frac{\mathrm{RR}_{UY|(A=0)} \times \mathrm{RR}_{SU|(A=0)}}{\mathrm{RR}_{UY|(A=0)} + \mathrm{RR}_{SU|(A=0)} - 1}$$

with the RR parameters defined as in section I.

Because the probabilities conditional on $S = 0$ aren't generally observed, we can replace those values with their possible extremes, 0 and 1, to obtain the bound:

$$\text{bias} \ \leq \mathrm{BF}_1 - P(Y = 1|A = 1, S = 1)/\mathrm{BF}_1 + P(Y = 1|A = 0, S = 1) \times \mathrm{BF}_0 \ . \tag{10}$$

2 When $S = U$

Result 2A. Bound

As in section I, when $S = U$, we can take the limit of equation (10) as each of the RR_{SU} terms in BF_1 and BF_0 approaches ∞:

$$\text{bias} \leq \lim_{\text{RR}_{SU}\to\infty} \text{BF}_1 - P(Y=1|A=1,S=1)/\text{BF}_1 + P(Y=1|A=0,S=1) \times \text{BF}_0$$

$$= \text{RR}_{UY|(A=1)} - P(Y=1|A=1,S=1)/\text{RR}_{UY|(A=1)} + P(Y=1|A=0,S=1) \times \text{RR}_{UY|(A=0)}$$

3 Increased risk in both exposure groups

Result 3A. Bound

When $P(Y=1|A=1,S=1) - P(Y=1|A=1,S=0)$ and $P(Y=1|A=0,S=1) - P(Y=1|A=0,S=0)$ are both greater than 0, (6) can be rewritten

$$\text{bias} \leq P(Y=1|A=1,S=1) - P(Y=1|A=1,S=0) \; .$$

Following the derivation of Result 2A, we find that

$$\text{bias} \leq P(Y=1|A=1,S=0) \times \frac{\text{RR}_{UY|(A=1)} \times \text{RR}_{SU|(A=1)}}{\text{RR}_{UY|(A=1)} + \text{RR}_{SU|(A=1)} - 1} - P(Y=1|A=1,S=1) \times \frac{\text{RR}_{UY|(A=1)} + \text{RR}_{SU|(A=1)} - 1}{\text{RR}_{UY|(A=1)} \times \text{RR}_{SU|(A=1)}} \; .$$

In terms of the observable data, we have:

$$\text{bias} \leq \frac{\text{RR}_{UY|(A=1)} \times \text{RR}_{SU|(A=1)}}{\text{RR}_{UY|(A=1)} + \text{RR}_{SU|(A=1)} - 1} - P(Y=1|A=1,S=1) \times \frac{\text{RR}_{UY|(A=1)} + \text{RR}_{SU|(A=1)} - 1}{\text{RR}_{UY|(A=1)} \times \text{RR}_{SU|(A=1)}} \; .$$

4 Increased risk in both exposure groups and when $S = U$

Result 4A. Bound

Combining Results 2A and 3A, we have that

$$\text{bias} \leq \text{RR}_{UY|(A=1)} - P(Y=1|A=1,S=1)/\text{RR}_{UY|(A=1)} \; .$$

5 Inference in the selected population

Result 5A. Bound

When we are concerned with the causal risk difference $P(Y_1 = 1|S=1) - P(Y_0 = 1|S=1)$, we assume that $Y_a \amalg A|\{S=1, U\}$; that is, conditioning on U is sufficient to eliminate the bias induced by conditioning on the selected population. This is an equivalent condition to that which requires U to suffice to control for confounding, conditional on measured confounders, in VanderWeele and Ding 2016b.[2] We can use their results for bounding the causal risk difference under unmeasured confounding as follows.

Define the following for arbitrary U with K levels (for notational simplicity, as in Section I):

$$\text{RD}^{\text{true}}_{AY^+|S=1} = P(Y=1|A=1,S=1) - \sum_{k=1}^{K} P(Y=1|A=0,S=1,U=k) P(U=k|A=1,S=1)$$

$$\text{RD}^{\text{true}}_{AY^-|S=1} = \sum_{k=1}^{K} P(Y=1|A=1,S=1,U=k)P(U=k|A=0,S=1) - P(Y=1|A=0,S=1)$$

$$\text{RD}^{\text{true}}_{AY|S=1} = P(A=1|S=1) \times \text{RD}^{\text{true}}_{AY^+|S=1} + (1 - P(A=1|S=1)) \times \text{RD}^{\text{true}}_{AY^-|S=1}$$

$$\text{bias } = \text{RD}^{\text{obs}}_{AY} - \text{RD}^{\text{true}}_{AY|S=1}$$

$$\text{BF}_U = \frac{\text{RR}_{UY|(S=1)} \times \text{RR}_{AU|(S=1)}}{\text{RR}_{UY|(S=1)} + \text{RR}_{AU|(S=1)} - 1}$$

where the parameters in BF_U are defined as in Section I and $\text{RD}^{\text{obs}}_{AY}$ as from Result 1A in Section II.

Because

$$\text{RD}^{\text{true}}_{AY|S=1} \geq \min\left(\text{RD}^{\text{true}}_{AY^+|S=1}, \text{RD}^{\text{true}}_{AY^-|S=1}\right) ,$$

we have that

$$\text{bias } \leq \max\left(\text{RD}^{\text{obs}}_{AY} - \text{RD}^{\text{true}}_{AY^+|S=1}, \text{RD}^{\text{obs}}_{AY} - \text{RD}^{\text{true}}_{AY^-|S=1}\right) .$$

Using the lower bounds for the causal risk differences from Ding and VanderWeele 2016b,[2] we have that

$$\text{RD}^{\text{true}}_{AY^+|S=1} - \text{RD}^{\text{obs}}_{AY} \leq P(Y=1|A=0,S=1) \times (\text{BF}_U - 1)$$

and

$$\text{RD}^{\text{true}}_{AY^-|S=1} - \text{RD}^{\text{obs}}_{AY} \leq P(Y=1|A=1,S=1) \times (1 - 1/\text{BF}_U) .$$

Therefore,

$$\text{bias } \leq \max\left(P(Y=1|A=0,S=1) \times (\text{BF}_U - 1), P(Y=1|A=1,S=1) \times (1 - 1/\text{BF}_U)\right) .$$

References

1. Ding P, VanderWeele TJ. Sharp sensitivity bounds for mediation under unmeasured mediator-outcome confounding. *Biometrika.* 2016;103:483–490.

2. Ding P, VanderWeele TJ. Sensitivity analysis without assumptions. *Epidemiology.* 2016;27:368–377.