To derive from a given ROC slope the corresponding pair of true positive rate (TPR) and false positive rate (FPR), the distributions of the diagnostic variable $x$ among diseased and non-diseased subgroups have to be estimated. We will herein consider the case for two normal distributions, $\mathcal{N}(\mu_1, \sigma_1^2)$ referring to the diseased and $\mathcal{N}(\mu_2, \sigma_2^2)$ referring to the non-diseased subgroup. For convenience, $\mu_1 > \mu_2$ is stipulated, with $\mu_1$ referring to the diseased subgroup.

ROC slope is defined for any $x$ as the likelihood-ratio at this point, such that:

$$ROC\text{ slope}(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}$$

That is, ROC slope is expressed as the ratio of two probability density functions. The optimal, cost-sensitive ROC slope at $x_c$ is given with $\frac{P(N)}{P(P)} \times \frac{(c_{fp}-c_{tn})}{(c_{fn}-c_{tp})}$, denoted by $R$ hereafter. It follows that (compare also for Pepe$^1$):

$$R = \frac{\sigma_2}{\sigma_1} \times e^{\frac{-(x_c-\mu_1)^2}{2\sigma_1^2} + \frac{(x_c-\mu_2)^2}{2\sigma_2^2}}$$

After application of the logarithm and some rearrangements, one is left with:

$$0 = x_c^2 \left( -\frac{1}{2\sigma_1^2} + \frac{1}{2\sigma_2^2} \right) + x_c \left( \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} \right) - \frac{\mu_1^2}{2\sigma_1^2} + \frac{\mu_2^2}{2\sigma_2^2} - \ln \left( \frac{\sigma_1}{\sigma_2} \times R \right)$$

This is of the form $0=ax^2+bx+c$, $x_c$ can therefore be obtained with the quadratic formula. If the variances of the two normally distributed variables are equal, the quadratic term is cancelled from the equation and there will be a unique solution for $x_c$. If the variances differ, there will be two solutions. That is, given $\sigma_1 \neq \sigma_2$, two different TPR-FPR pairs will present as the optimal classifier for a given $\rho$ and disease prevalence.

For the example chosen ($\sigma_1$, $\sigma_2$ and $\mu_1$ set to 1 and $\mu_2$ to -1), the last equation reduces to $x_c=0.5 \times \ln(R)$. For each stipulated disease prevalence and value of the fraction $\rho=(c_{fp}-c_{tn})/(c_{fn}-c_{tp})$, a value $x_c$ can be derived. $x_c$ is the diagnostic threshold of variable $x$, values $>x_c$ denote a positive classifier result. TPR and FPR now can be derived by their respective cumulative distribution functions, in the example: $TPR=1-\Phi(x_c-\mu_1)=\Phi(\mu_2-x_c)$ and analogously for FPR.